

Hyperchaos and Julia Sets

O. E. Rössler and C. Kahlert

Institute for Physical and Theoretical Chemistry, University of Tübingen, West Germany

J. Parisi, J. Peinke, and B. Röhricht

Physical Institute, University of Tübingen, West Germany

Z. Naturforsch. **41a**, 819–822 (1986); received March 10, 1986

Julia sets are self-similar separatrices of the coast-line type found in noninvertible 2-D maps. The same class of maps also generates hyperchaos (chaos with two mixing directions). Smale's notion of a "nontrivial basic set" provides a connection. These sets arise when a chaotic (or hyperchaotic) attractor "explodes". In the case of more than one escape route, this set becomes a "fuzzy boundary" (Mira). Its projection as the map becomes noninvertible (1-D) is a "Julia set in 1 D". In the analogous hyperchaos case the 2-D limiting map contains a classical Julia set of the continuous type. An identically looking set can also be obtained within a non-exploded hyperchaotic attractor, however, as a "cloud". Julia-like attractors therefore exist. The theory also predicts Mandelbrot sets for 4-D flows. Julia-like behavior is a new, numerically easy-to-test-for property of most nontrivial dynamical systems.

Both hyperchaos and Julia sets occur in the same class of dynamical systems. Julia sets occur in 2-D noninvertible maps [1]; so does hyperchaos [2–4]. Julia sets are beautiful [5–7] chaotic [8] sets; so is hyperchaos (cf. the picture in [4]). There must be a relation.

There also must be a connection of Julia sets to continuous systems. All noninvertible maps possess noninvertible continuous flows as suspensions (cf. [9]). Hence Julia sets exist as separatrices in certain continuous systems at least. Their "neighboring structures" in ordinary differentiable systems will be worth exploring – for example, in the explicit map and flow in [4].

An alternative, qualitative approach takes the above relation seriously. Is a nontrivial separating structure perhaps already known to exist in invertible systems in connection with a hyperchaotic attractor?

The answer is yes. Indeed one of the earliest notions in global analysis is that of a nontrivial basic set [10]. Smale introduced it because it exhibits the major properties of a strange attractor without actually being one. (At that time only a higher-dimensional example, the solenoid [10], was known.) A nontrivial basic set is a generalized

saddle (the trivial basic set). It consists of all the uncountably many unstable periodic solutions of up to countable periodicity (a Cantor set) that characterize a chaotic attractor, but lacks the finite-measure bulk of nonperiodic solutions of the latter. It could be called its skeleton (cf. [11]). The relation indeed is very easy to see. Smale found his set in a folded-over map ("horseshoe map") whose middle portion of interest could be assumed linear. Soon after, chaotic attractors were discovered in flows whose cross section looked like an asymmetric horseshoe (walking stick) [12]. The same holds true for Hénon's [13] explicit 2-D diffeomorphism (forgetting about its nonorientability). These systems at first, at little "overlap" of the walking stick's handle, exhibit a simple period-1 attractor (with period-1 transients) – a Hopf limit cycle (see the computer movie described in [14]). Then, as the overlap is increased, this attractor acquires period-2 transients as it comes to lie in the folded-over region [15]. Soon thereafter, through a series of period-doublings, a chaotic attractor arises [14]. Only with a further increase in overlap can the attractor be brought to "exploding" [14]. Hereby the folding-over has actually intruded into a neighboring basin of attraction by its crossing a vertical separatrix in the "foot" of the walking stick. The surviving set of periodic solutions (who by this very property are shielded from being captured) is the Smale basic set.

Reprint requests to O. E. Rössler, Institut für Physikalische und Theoretische Chemie, Universität Tübingen, Auf der Morgenstelle 8, 7400 Tübingen 1.

0340-4811 / 86 / 0600-0819 \$ 01.30/0. – Please order a reprint rather than making your own copy.



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition "no derivative works"). This is to allow reuse in the area of future scientific usage.

This well-studied transition (cf. [9]) reflects a general principle. Whenever a chaotic or hyperchaotic attractor is punctured, its skeleton of periodic solutions remains in existence as its “corresponding” basic set. This basic set already in the simplest case is a product of two Cantor sets. In the one projection (a piecewise-linear 1-D map in the case of the horseshoe map), all the “middle thirds” are being “bitten out” in positive time. In the full picture, everything at the same time gets compressed in a direction orthogonal to the former, generating the usual “Cantor set of lines” structure of a chaotic attractor. The two rarifying processes cooperate, leaving only a generalized Cantor set of points in the limit (cf. [10]). A Cantor set where “middle Swiss flags” are being removed consecutively from a square, rather than “middle thirds” from a line, was apparently first considered by Alexandroff [16].

Smale [10] considered only the simplest possibility – with just one neighboring basin protruding into the region of overlap. Here the basic set “separates” from the new basin nothing but its (the basic set’s) own points plus their associated stable manifolds (insets [17]). The basic set in this case nevertheless is an analogue to a discontinuous Julia set already.

A generalization to the case of two puncturings is fairly straightforward. The skeleton within the Alexandroff cross is the same. But this time it is either adjacent basin (either color) that is biting smaller and smaller partial flags out. The remaining “web” of points is everywhere adjacent to both colors. It thus has the major phenomenological properties of a continuous Julia set (cf. [7]). At the same time we know it is a Smale basic set (stable manifolds added).

Many details remain to be worked out. (Must there be a “cycle” in the sense of Smale [10] involved, and hence infinitely many periodic attractors [18]? How about three colors?) The first example of a boundary of the present type appears to have been discovered by Mira [19] in a study of two different embedded periodic attractors coexisting in a cubic analogue to Hénon’s map [13]. He chose the name “fuzzy boundary” (*frontière floue*) to characterize the object. Both basins were found to accumulate on a Cantor set.

In order to arrive at a classical Julia set, 2-D diffeomorphisms are not appropriate since they are

invertible. Moreover the present boundary is of saddle-type and thus not purely repelling like Julia’s [1, 8]. Both properties can be amended by going to the noninvertible (1-D) limit. Here one looks only at the “top” of the multiply folded snake; the Alexandroff cross degenerates to an ordinary Cantor set. The distribution of red and blue islands nevertheless remains most complicated. An appropriate name for the resulting object appears to be “classical Julia set in 1-D”. These objects were again first seen by Gumowski and Mira [20]. Interestingly, these authors (cf. [9], p. 393) already saw that with any such object there exists an analogous object in parameter space. Since (by the Smale construction) a purely passively coupled second variable can always be added without changing the object, it is even possible to predict that the whole 2-D structure mentioned above should be found again in parameter space. This is indeed apparently the case [21].

To recapture a classical continuous Julia set, in 2-D, one has to proceed to the next higher dimension, however. Here in the invertible (3-D) case, the multiply punctured hyperchaotic attractor gives rise to a basic set that is an “Alexandroff hypercross” in the simplest case. Its associated continuous boundary, if there is more than one color, is even harder to characterize than in the preceding case, however, because the (still one-dimensional) stable manifolds of the periodic points do not readily lend themselves to the formation of a closed curtain this time. The open questions multiply. However, going to the noninvertible (2-D) limit acts as a simplification again. It is here that we obtain a classical continuous Julia set for the first time. The simplest – almost trivial – example is that of two uncoupled logistic maps (or tent maps) in the “exploded” state. A full-fledged Alexandroff cross is generated in case the vertical and horizontal stripes taken out are assumed to have the same color. Introducing a different label for the two basins shows that each cross consists of two differently colored bars – but such that in the middle there is a letter-X shaped division. Thus two red pencils and two blue ones meet point-to-point in the middle of each cross [22] (cf. below). It will be fun to study the canonical transformations of this “simplest” continuous Julia set. Metzler’s set, discovered empirically only recently in two cross-activatingly coupled logistic maps [23], ought to be among them. Moreover, the above derivation shows that all

potentially hyperchaotic systems ought to possess this level of complexity in their parameter spaces. A first example in an invertible ODE has just been found [24].

Finally, a glimpse at the next higher dimension. Unexpectedly, the single negative result reported above (lack of a purely repelling continuation of a Julia set into neighboring invertible systems; so that still no “cloud shaped” attractor – or repeller – exists in ordinary dynamics) appears to be non-final. This follows from a reconsideration of the above cross. It was easy to get by cutting a hole (in fact, two) into a folded towel as we saw. But does this “cutting” indeed have to be final? The answer is no. Indeed, any labeling will do. Taking trajectories out in reality is no different mathematically than using a gentler way to mark them. Both procedures end up in a “color coded symbolic dynamics” [22], cf. Fig. 1. Now this purely formal step can be taken seriously again dynamically – by simply making an additional variable dependent on the attractor’s having hit a certain subarea of itself or not. As soon as this is the case, we have an analogue to a real cloud in our ODE. There too a hyperchaotic (turbulent) attractor becomes labeled – by suspended particles responding somehow to a particular substate having been reached locally (like a certain temperature). This “labeling”, involving a variable of its own, then has secondary consequences again (which, however, need not undo the generating process just described). This basic idea has yet to be translated into a working 5-D flow or 4-D invertible map – or 3-D *noninvertible* map.

To conclude, bringing Julia sets back into the fold of ordinary dynamics promises some unexpected vistas. Some of the predictions arrived at could already be confirmed [21, 22, 24]. Others undoubtedly will follow suit – some with negative results. The picture obtained so far warrants a first sweeping verdict. A new universal property of most natural dynamical systems with potentially interesting behavior is “Julia like behavior in both state space

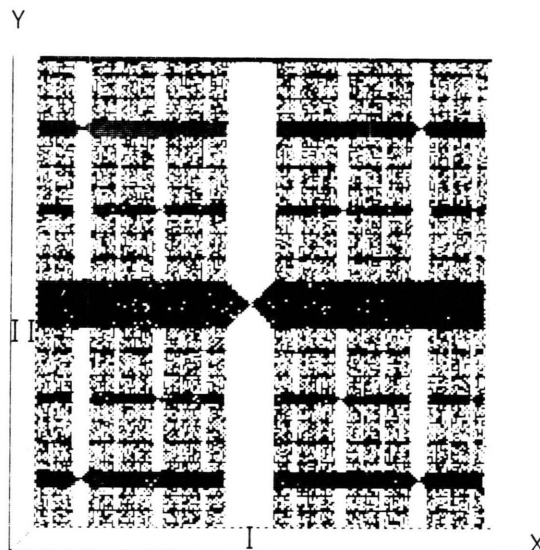


Fig. 1. Simplest Julia-like set. Eq.: $x_{n+1} = 3.99 x_n (1 - x_n)$, $y_{n+1} = 3.99 y_n (1 - y_n)$. Initial conditions: $x_1 = 0.2$, $y_1 = 0.2 + 10^{-11}$. Axes: 0 to 1 for x and y . Plotting convention: Those trajectorial segments landing next time in the region marked I (II) are shown in black (white). Number of iterates shown: $3 \cdot 10^6$. By letting the system run longer, a sharp boundary between black and white will be produced eventually at all magnifications. Alternatively, it is also possible to systematically scan all initial conditions along a grid of finite resolution (the usual technique [5] for finding boundaries). Note that in the latter case, having two pseudo sinks in the hyperchaos, as assumed here, no longer has any advantage over real sinks. Indeed, on changing the parameter 3.99 to 4.04 in the above equations, so that two genuine escape hatches arise (for $y < 0$ and $x < 0$, respectively), the same picture is re-obtained except for a small change in size – as a genuine Julia set. Cf. text above. From [22].

and parameter space”. It would be nice to have a look at the catalogue of finitely many prototype “finger-prints” against which every new nonlinear system will have to be checked routinely in a few years to come, today.

Acknowledgements

Work supported in part by the DFG. O. E. R. thanks Michael Michelitsch for a discussion in September 1985.

- [1] G. Julia, *J. Math. Pures et Appl.* **4**, 47 (1918).
- [2] O. E. Rössler, *Phys. Lett.* **71 A**, 155 (1979).
- [3] O. E. Rössler and C. Mira, Higher-order Chaos in a Constrained Differential Equation with an Explicit Cross Section, Mathematisches Forschungsinstitut Oberwolfach, Tagungsberichte, Sept. 13–19, 1981.
- [4] O. E. Rössler, *Z. Naturforsch.* **38a**, 788 (1983).

- [5] B. Mandelbrot, *Ann. N.Y. Acad. Sci.* **357**, 249 (1980).
- [6] A. Douady and J. H. Hubbard, *C.R. Acad. Sci. Paris* **294**, 123 (1982).
- [7] H. O. Peitgen and P. H. Richter, *Morphology of Complex Boundaries* (in German). Exhibition Catalogue, University of Bremen (1984); *Ber. Bunsenges. Phys. Chem.* **89**, 571 (1985); *Phys. Bl.* **42**, 9 (1986).

- [8] H. A. Lauwerier, Two-dimensional iterative maps, in: *Chaos* (A. V. Holden, ed.), Manchester University Press, Manchester 1986, pp. 58–95.
- [9] I. Gumowski and C. Mira, *Dynamique Chaotique, Transformations Ponctuelles, Transition Ordre-Désordre*, Cepadues Editions, Toulouse 1980.
- [10] S. Smale, *Bull. Amer. Math. Soc.* **73**, 747 (1967).
- [11] O. E. Rössler, Long Line Attractors, in: *Iteration Theory and Its Functional Equations* (R. Liedl, L. Reich, and Gy. Targonski, eds.), Springer Lect. Notes Math. **1163**, 149 (1985).
- [12] O. E. Rössler, *Z. Naturforsch.* **31a**, 259 (1976).
- [13] M. Hénon, *Comm. Math. Phys.* **50**, 69 (1976).
- [14] O. E. Rössler, *Z. Naturforsch.* **31a**, 1664 (1976).
- [15] O. E. Rössler, *Z. Naturforsch.* **32a**, 607 (1977).
- [16] P. S. Alexandroff, *Introduction to Set Theory and the Theory of Real Functions* (in Russian). Ogie, Moscow 1948.
- [17] R. H. Abraham and C. D. Shaw, *Dynamics – The Geometry of Behavior*, Vol. 3: *Global Behavior*. Aerial Press, Santa Cruz 1985.
- [18] S. E. Newhouse and J. Palis, *Astérisque* **31**, 43 (1976).
- [19] C. Mira, *C.R. Acad. Sci. Paris A* **288**, 591 (1979).
- [20] I. Gumowski and C. Mira, *C.R. Acad. Sci. Paris A* **280**, 905 (1975).
- [21] C. Kahlert and O. E. Rössler, Self-similar set in the parameter space of a 1-D map (letter-*N* map), to be submitted.
- [22] O. E. Rössler, *Clouds in Flows*, to be submitted.
- [23] W. Metzler, The Route to Chaos of Two Coupled Logistic Maps, Paper presented at “Dynamics Days”, La Jolla, January 7–10, 1986.
- [24] B. Röhricht *et al.*, Search for a Mandelbrot set in a 4-D flow (two-compartment Turing oscillator), to be submitted; cf. also C. Kahlert *et al.*, self-similarity in two directions in the parameter space of a piecewise-linear 4-variable ODE, to be submitted.